AP Calculus BC

Topic: Logistic Growth

Instructions

Solve the following problems. Show all your work clearly. Include constants and coefficients for partial fractions, and evaluate integrals where required.

Practice Problems

1. **Partial Fraction Decomposition:** Find the values of A and B for the following rational expressions:
   (i) $\frac{x-10}{x^2-3x} = \frac{A}{x} + \frac{B}{x-3}$
   (ii) $\frac{5x+8}{x^2+4x-5} = \frac{A}{x-1} + \frac{B}{x+5}$
   (iii) $\frac{4x-3}{x^2-9} = \frac{A}{x-3} + \frac{B}{x+3}$
2. **Evaluate the Integral:** Evaluate the following integrals using partial fractions:
   (i) $\int \frac{3x+2}{x^2-5x+6}dx$
   (ii) $\int \frac{2x^2}{x^3-9x}dx$
   (iii) $\int \frac{6}{x^2-1}dx$
3. **Solve the Differential Equation:** Solve the following differential equations:
   (i) $\frac{dy}{dx} = \frac{4x}{x^2-3x}$
   (ii) $F'(x) = \frac{2x}{x^2+x-6}$
   (iii) $G'(t) = \frac{t^2}{t^3-4t}$

4. **Integrate Without Partial Fractions:** Find the integral directly (without decomposing into partial fractions):

   (i) $\int \frac{x^2+3}{x^2-x}dx$

   (ii) $\int \frac{3x^2}{x^3-x^2}dx$

   (iii) $\int \frac{x^2+2x+1}{x^2-3x+2}dx$

5. **Basic Logistic Growth Model:** Solve the following differential equations for logistic growth:

   (i) $\frac{dP}{dt} = 0.2P(1 - \frac{P}{50})$, with $P(0) = 5$.

   (ii) $\frac{dP}{dt} = 0.1P(1 - \frac{P}{100})$, with $P(0) = 10$.

   (iii) $\frac{dN}{dt} = 0.3N(1 - \frac{N}{300})$, with $N(0) = 20$.

   (iv) $\frac{dP}{dt} = 0.05P(1 - \frac{P}{200})$, with $P(0) = 50$.

6. **Real-World Applications:**

   (i) A population of fish in a lake grows according to the logistic equation:

   $$\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{500}\right),$$

   where P is the population size and t is time in years. If the initial population is 50, find the population size after 10 years.

   (ii) A bacterial culture grows according to the logistic equation:

   $$\frac{dP}{dt} = 0.5P\left(1 - \frac{P}{1000}\right).$$

   If the initial population is 200, determine the limiting population size.

Multiple Choice Questions

1. Which of the following is the correct partial fraction decomposition of $\frac{3x+7}{x^2-4}$?

   a. $\frac{A}{x+2} + \frac{B}{x-2}$

   b. $\frac{A}{x-2} + \frac{B}{x^2-2}$

   c. $\frac{A}{x} + \frac{B}{x^2-4}$

   d. $\frac{A}{x-1} + \frac{B}{x+2}$

2. Evaluate $\int \frac{4x}{x^2-1}dx$ using partial fractions.

   a. $\ln|x+1| - \ln|x-1| + C$

   b. $\ln|x-1| + \ln|x+1| + C$

   c. $2\ln|x-1| + C$

   d. $2\ln|x+1| + C$

3. Which of the following represents the general solution of the logistic equation $\frac{dP}{dt} = kP(1 - \frac{P}{K})$?

   a. $P(t) = \frac{K}{1+Ce^{-kt}}$

   b. $P(t) = \frac{K}{1-Ce^{kt}}$

   c. $P(t) = \frac{K}{1+Ce^{kt}}$

   d. $P(t) = \frac{K}{1-Ce^{-kt}}$

4. Solve $\int \frac{6}{x^2-9} dx$ using partial fractions.

   a. $\ln|x+3| - \ln|x-3| + C$

   b. $\frac{1}{3}\ln|x+3| + \ln|x-3| + C$

   c. $\ln|x-3| + \ln|x+3| + C$

   d. $\frac{1}{2}\ln|x+3| + \frac{1}{2}\ln|x-3| + C$