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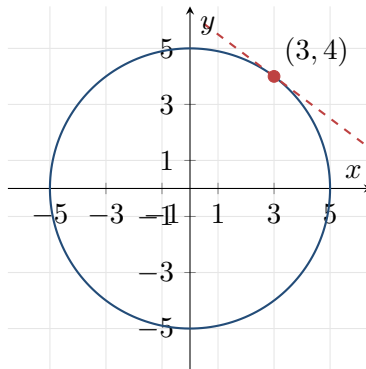
IMPLICIT DIFFERENTIATION

AP Calculus AB Practice Worksheet

Learning Goal: Differentiate equations that define y implicitly as a function of x , find slopes and equations of tangent lines on curves, locate horizontal and vertical tangents, and compute second derivatives implicitly.

1. Explicit vs. Implicit

The circle $x^2 + y^2 = 25$ is not the graph of a single function, but it can be split into two functions.



1. Solve $x^2 + y^2 = 25$ for y . Write the two functions whose graphs form the upper and lower halves of the circle.

2. Differentiate the upper-half function *explicitly* to find the slope of the tangent line at $(3, 4)$.

3. What would make this method difficult for an equation like $x^3 + y^3 = 6xy$?

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2. The Method of Implicit Differentiation

1. Differentiate both sides of the equation with respect to x .
2. Treat y as a function of x : every y -term gets a factor of $\frac{dy}{dx}$ (chain rule).
3. Collect all terms containing $\frac{dy}{dx}$ on one side.
4. Factor out $\frac{dy}{dx}$ and solve.

Key idea: $\frac{d}{dx}(y^2) = 2y \frac{dy}{dx}$, not just $2y$.

3. Warm-Up: Differentiate Each Term

Assume y is a differentiable function of x . Find each derivative.

Expression	$\frac{d}{dx}$	Expression	$\frac{d}{dx}$
x^3		y^3	
$5y$		$\sqrt{y}$	
xy		x^2y^3	
$\sin y$		e^y	
$\ln y$		$\cos(xy)$	
$\frac{x}{y}$		$\tan^{-1} y$	

Explain in your own words why $\frac{d}{dx}(x^3)$ and $\frac{d}{dx}(y^3)$ look different.

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4. Find dy/dx : Basic Curves

Use implicit differentiation to find $\frac{dy}{dx}$.

1.

$$x^2 + y^2 = 25$$

2.

$$x^3 + y^3 = 9$$

3.

$$4x^2 - 9y^2 = 36$$

4.

$$y^2 = x^3 - 2x$$

5.

$$\sqrt{x} + \sqrt{y} = 4$$

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5. Find dy/dx : Product and Quotient Terms

Use implicit differentiation to find $\frac{dy}{dx}$. Watch for products of x and y .

1.

$$xy = 6$$

2.

$$x^2y + y^3 = 10$$

3.

$$x^3 - xy + y^2 = 7$$

4.

$$\frac{x}{y} + y = x^2$$

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6. Find dy/dx : Trig, Exponential, and Log Curves

Use implicit differentiation to find $\frac{dy}{dx}$.

1.

$$\sin y = x$$

2.

$$y = \sin(x + y)$$

3.

$$e^{xy} = x + y$$

4.

$$\ln(xy) = x + y \quad (\text{Hint: rewrite with log properties first.})$$

5.

$$\tan y = x$$

Then write $\frac{dy}{dx}$ in terms of x only.

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7. Slope at a Point

Find $\frac{dy}{dx}$ at the given point. First check that the point lies on the curve.

1.

$$x^2 + xy + y^2 = 7, \quad (1, 2)$$

2.

$$y^2 - x^2 = 3, \quad (1, 2)$$

3.

$$x^2y^2 = 2x + y, \quad (1, 2)$$

4.

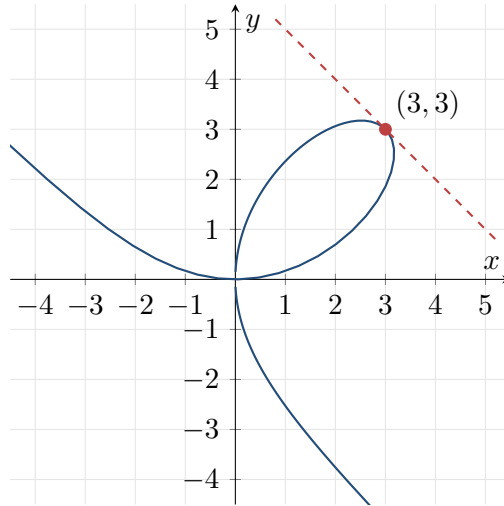
$$\sin(\pi y) + x^2 = 1, \quad (1, 0)$$

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8. Tangent and Normal Lines

The *folium of Descartes*, $x^3 + y^3 = 6xy$, is shown below with its tangent line (dashed) at $(3, 3)$.



- Find $\frac{dy}{dx}$ for $x^3 + y^3 = 6xy$. Then find the equation of the tangent line and the normal line at $(3, 3)$. Does your tangent line match the graph?

- Write equations of the tangent and normal lines to $x^2 + y^2 = 25$ at $(3, -4)$.

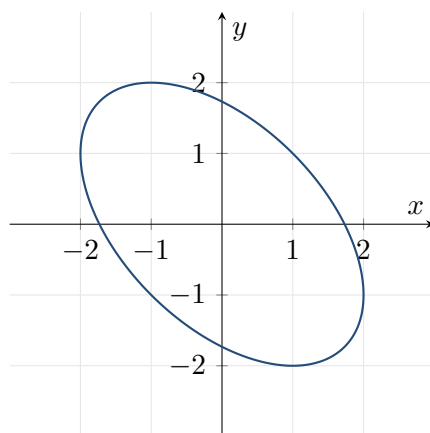
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9. Horizontal and Vertical Tangent Lines

If $\frac{dy}{dx} = \frac{N}{D}$, then the tangent line is **horizontal** where $N = 0$ (and $D \neq 0$), and **vertical** where $D = 0$ (and $N \neq 0$). Always substitute back into the *original* equation to find the points.

The curve $x^2 + xy + y^2 = 3$ is a tilted ellipse.



1. Show that $\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$.

2. Find all points where the tangent line is horizontal. Mark them on the graph.

3. Find all points where the tangent line is vertical. Mark them on the graph.

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10. More Horizontal and Vertical Tangents

1. For the circle

$$(x - 1)^2 + (y + 2)^2 = 9,$$

find all points with a horizontal tangent and all points with a vertical tangent. Check that your answers make sense geometrically.

2. For the curve

$$y^2 = x^3 - 3x + 3,$$

find all points where the tangent line is horizontal.

11. Second Derivatives Implicitly

To find $\frac{d^2y}{dx^2}$: differentiate $\frac{dy}{dx}$ again (quotient rule), then substitute your expression for $\frac{dy}{dx}$ and simplify using the original equation.

1. For
- $x^2 + y^2 = 25$
- , show that
- $\frac{d^2y}{dx^2} = -\frac{25}{y^3}$
- .

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12. More Second Derivatives

Find $\frac{d^2y}{dx^2}$ in terms of x and y .

1.

$$xy = 4$$

2.

$$y^2 = x^2 + 4$$

3.

$$x^3 + y^3 = 1$$

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13. Proving Derivative Rules Implicitly

1. **Rational powers.** Let $y = x^{p/q}$, so $y^q = x^p$. Use implicit differentiation to show that

$$\frac{dy}{dx} = \frac{p}{q} x^{(p/q)-1}.$$

2. **Natural logarithm.** Let $y = \ln x$, so $e^y = x$. Use implicit differentiation to show that $\frac{dy}{dx} = \frac{1}{x}$.

14. Explain Your Thinking

1. A student differentiates $x^2 + y^2 = 25$ and writes $2x + 2y = 0$. Explain the mistake and correct it.

2. The expression for $\frac{dy}{dx}$ from implicit differentiation usually contains both x and y . Why do we need a *point* (x, y) , not just an x -value, to find a slope?

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15. AP-Style Problem

Consider the curve given by

$$y^3 - xy = 2.$$

1. Show that $\frac{dy}{dx} = \frac{y}{3y^2 - x}$.

2. Find the y -coordinate of the point on the curve where $x = -1$. Then write an equation of the tangent line at that point.

3. Find the coordinates of the point on the curve at which the tangent line is vertical.

4. Find $\frac{d^2y}{dx^2}$ at the point found in part (b).

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IMPLICIT DIFFERENTIATION

Mixed Review

16. Mixed Practice

Show your work clearly.

1. Find $\frac{dy}{dx}$ for $x^2 - 3xy + y^2 = 1$.

2. Find $\frac{dy}{dx}$ for $y \cos x = x^2 + y^2$.

3. Find the slope of $x^{2/3} + y^{2/3} = 5$ at the point $(8, 1)$.

4. Find $\frac{dy}{dx}$ for $e^y = x^2 + y$, then find where the tangent is vertical.

17. Challenge Problem

Show that the curves $x^2 - y^2 = 5$ and $4x^2 + 9y^2 = 72$ intersect at right angles at the point $(3, 2)$. (Two curves are *orthogonal* at a point if their tangent lines there are perpendicular.)

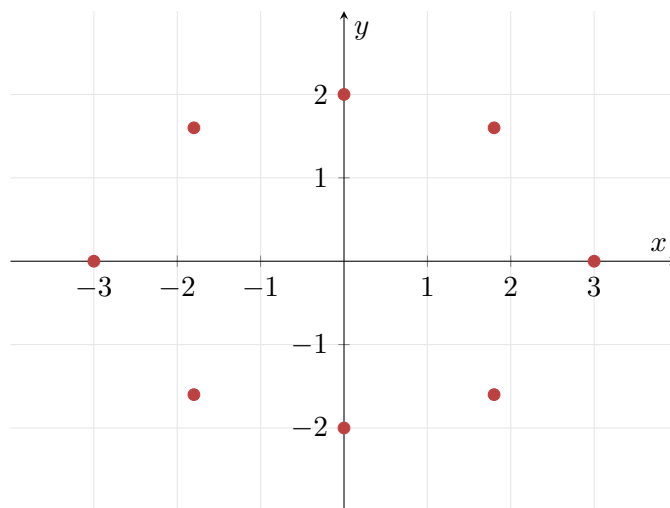
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18. Graphing Challenge

For the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$:

- Find $\frac{dy}{dx}$ by implicit differentiation.
- At each marked point, compute the slope and draw a short tangent segment.
- Use your tangent segments to sketch the ellipse.



Point	(1.8, 1.6)	(-1.8, 1.6)	(0, 2)	(3, 0)
Slope				

Reflection: What is one thing you understand better about implicit differentiation after completing this worksheet?
