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DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

AP Calculus AB Practice Worksheet

Learning Goal: Derive and apply the derivative formulas for the inverse trigonometric functions, combine them with the chain, product, and quotient rules, find derivatives of inverse functions at a point, and solve tangent-line and rate-of-change problems.

1. Inverse Trigonometric Functions Review

Complete the table. Each inverse function is defined on a restricted interval so that it is one-to-one.

Function	Domain	Range
$y = \sin^{-1} x$		
$y = \cos^{-1} x$		
$y = \tan^{-1} x$		
$y = \sec^{-1} x$		
$y = \cot^{-1} x$		

2. Evaluate Inverse Trigonometric Values

Give exact values in radians.

1. $\sin^{-1}\left(\frac{1}{2}\right) =$ _____ $\cos^{-1}\left(-\frac{1}{2}\right) =$ _____

2. $\tan^{-1}(\sqrt{3}) =$ _____ $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) =$ _____

3. $\sec^{-1}(2) =$ _____ $\tan^{-1}(-1) =$ _____

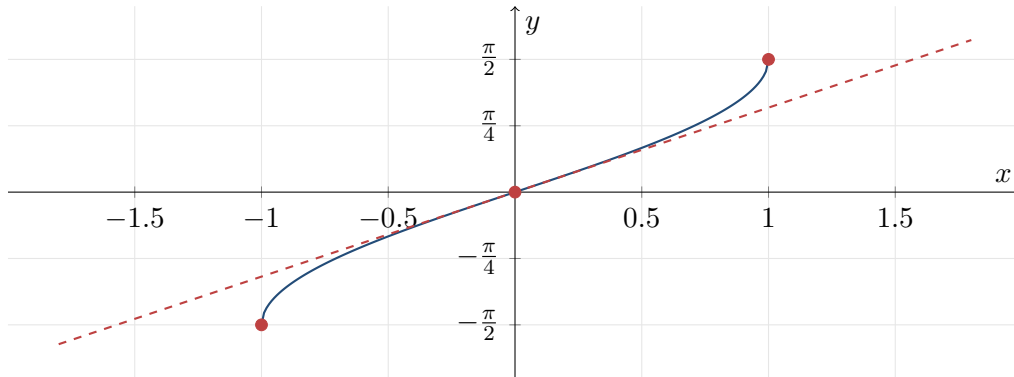
4. Explain why $\sin^{-1}(2)$ is undefined.

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3. Read the Slope from the Graph

The graph of $y = \sin^{-1} x$ is shown below, together with its tangent line (dashed) at the origin.



Answer the following:

1. Use the tangent line to find the slope of $y = \sin^{-1} x$ at $x = 0$.
2. Is the slope of $y = \sin^{-1} x$ ever negative? Explain using the graph.

3. What happens to the steepness of the graph as $x \rightarrow 1^-$ and as $x \rightarrow -1^+$?

4. Based on the graph, is $y = \sin^{-1} x$ differentiable at $x = 1$? Explain.

5. Sketch the tangent line to the graph at $x = 0.5$ and estimate its slope. Later, compare your estimate with the exact value you find in Section 12.

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4. Derive the Derivative of $\sin^{-1} x$

Use implicit differentiation and the right triangle to fill in each blank.

Let $y = \sin^{-1} x$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

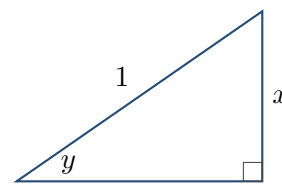
Then $\sin y =$ _____

Differentiate: _____ $\cdot \frac{dy}{dx} = 1$

Solve: $\frac{dy}{dx} =$ _____

From the triangle: $\cos y =$ _____

Therefore: $\frac{d}{dx} (\sin^{-1} x) =$ _____

Why do we choose the *positive* square root for $\cos y$? (Hint: what is the range of y ?)

Label the missing side.

5. Your Turn: Derive the Derivative of $\tan^{-1} x$ Let $y = \tan^{-1} x$. Use $\tan y = x$, implicit differentiation, and the identity $\sec^2 y = 1 + \tan^2 y$ to show that

$$\frac{d}{dx} (\tan^{-1} x) = \frac{1}{1 + x^2}.$$

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6. Derivative Rules

Let u be a differentiable function of x .

$$\frac{d}{dx} (\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad |u| < 1$$

$$\frac{d}{dx} (\cos^{-1} u) = -\frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad |u| < 1$$

$$\frac{d}{dx} (\tan^{-1} u) = \frac{1}{1+u^2} \frac{du}{dx}$$

$$\frac{d}{dx} (\cot^{-1} u) = -\frac{1}{1+u^2} \frac{du}{dx}$$

$$\frac{d}{dx} (\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2-1}} \frac{du}{dx}, \quad |u| > 1$$

$$\frac{d}{dx} (\csc^{-1} u) = -\frac{1}{|u|\sqrt{u^2-1}} \frac{du}{dx}, \quad |u| > 1$$

Pattern: Each *co*-function derivative is the negative of its partner, because

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x, \quad \cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x, \quad \csc^{-1} x = \frac{\pi}{2} - \sec^{-1} x.$$

7. Explain Your Thinking

Use the identity $\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$ to explain why

$$\frac{d}{dx} (\cos^{-1} x) = -\frac{d}{dx} (\sin^{-1} x).$$

What does this tell you about the graph of $y = \cos^{-1} x$: is it increasing or decreasing on $(-1, 1)$?

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8. Derivatives with $\sin^{-1}$ and $\cos^{-1}$ Find $\frac{dy}{dx}$. Simplify where possible.

1.

$$y = \sin^{-1}(3x)$$

2.

$$y = \cos^{-1}(x^2)$$

3.

$$y = \sin^{-1}(\sqrt{x})$$

4.

$$y = x \sin^{-1} x + \sqrt{1 - x^2}$$

5.

$$y = \cos^{-1}(e^x)$$

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9. Derivatives with $\tan^{-1}$ and $\cot^{-1}$

Find $\frac{dy}{dx}$. Simplify where possible.

1.

$$y = \tan^{-1}(5x)$$

2.

$$y = \tan^{-1}\left(\frac{x}{2}\right)$$

3.

$$y = x^2 \tan^{-1} x$$

4.

$$y = \cot^{-1}\left(\frac{1}{x}\right)$$

5.

$$y = \tan^{-1}(\ln x)$$

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10. Derivatives with $\sec^{-1}$ and $\csc^{-1}$

Find $\frac{dy}{dx}$. Pay attention to the absolute value.

1.

$$y = \sec^{-1}(4x), \quad x > \frac{1}{4}$$

2.

$$y = \csc^{-1}(x^2 + 1), \quad x \neq 0$$

3.

$$y = \sec^{-1}(e^x), \quad x > 0$$

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11. Combine the Rules

Find $\frac{dy}{dx}$. You will need more than one rule.

1.

$$y = \frac{\sin^{-1} x}{x}$$

2.

$$y = (\tan^{-1} x)^3$$

3.

$$y = \sin^{-1} x \cdot \cos^{-1} x$$

4.

$$y = \tan^{-1} \left(\frac{1+x}{1-x} \right), \quad x \neq 1$$

Simplify completely. What do you notice?

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12. Evaluate the Derivative at a Point

Find the exact value. Simplify radicals.

1. If $f(x) = \sin^{-1} x$, find $f'(\frac{1}{2})$.

2. If $g(x) = \tan^{-1}(x^2)$, find $g'(1)$.

3. If $h(x) = x \cos^{-1} x$, find $h'(0)$.

4. If $p(x) = \cos^{-1}(2x)$, find $p'(\frac{1}{4})$.

5. If $q(x) = \tan^{-1}(e^x)$, find $q'(0)$.

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13. Equations of Tangent Lines

Write an equation of the tangent line at the given x -value.

1.

$$y = \tan^{-1} x, \quad x = 1$$

2.

$$y = \sin^{-1}(2x), \quad x = \frac{1}{4}$$

3.

$$y = \sec^{-1} x, \quad x = 2$$

4. Write an equation of the *normal* line (perpendicular to the tangent line) to

$$y = \tan^{-1}(3x)$$

at the origin.

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14. Derivatives of Inverse Functions

If f is differentiable and one-to-one, and $f'(f^{-1}(a)) \neq 0$, then

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))} \quad \text{or} \quad \frac{dx}{dy} = \frac{1}{dy/dx}$$

1. Let

$$f(x) = x^3 + x + 1.$$

Find $(f^{-1})'(3)$. (Hint: first find x so that $f(x) = 3$.)

2. Let

$$f(x) = 2x + \cos x.$$

Find $(f^{-1})'(1)$.

3. The table gives values of a one-to-one differentiable function g and its derivative.

x	1	2	3	4
$g(x)$	3	5	8	12
$g'(x)$	2	$\frac{5}{2}$	4	6

Let $h = g^{-1}$. Find $h(5)$, $h'(5)$, and $h'(8)$.

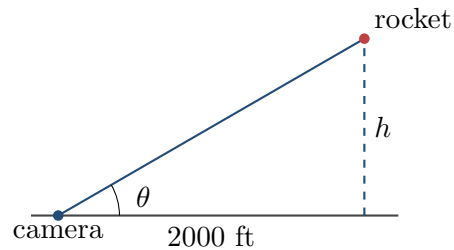
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15. Rates of Change in Context

1. A camera on the ground is 2000 feet from a rocket's launch pad. When the rocket is h feet high, the camera's angle of elevation is

$$\theta = \tan^{-1}\left(\frac{h}{2000}\right).$$



Find $\frac{d\theta}{dh}$ when $h = 2000$ ft. Include units, and explain what your answer means.

2. As the rocket climbs higher, does θ change faster or slower for each additional foot of height? Use $\frac{d\theta}{dh}$ to justify your answer.

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16. A Surprising DerivativeFor $x > 0$, let

$$F(x) = \tan^{-1} x + \tan^{-1} \left(\frac{1}{x} \right).$$

1. Find $F'(x)$ and simplify.

2. What does your answer tell you about F ? Evaluate $F(1)$ to find the exact value of $F(x)$ for all $x > 0$.

17. Prove an Identity with Derivatives

Let

$$G(x) = \sin^{-1} x + \cos^{-1} x, \quad -1 < x < 1.$$

1. Show that $G'(x) = 0$.

2. Evaluate $G(0)$. What identity have you proved?

3. Explain why a function whose derivative is 0 on an interval must be constant on that interval.

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INVERSE TRIGONOMETRIC DERIVATIVES

Mixed Review

18. Mixed Practice

Show your work clearly.

1. Find $\frac{dy}{dx}$ for $y = \sin^{-1}\left(\frac{x}{3}\right)$.

2. Find $\frac{dy}{dx}$ for $y = e^x \tan^{-1} x$.

3. Find $\frac{dy}{dx}$ for $y = \ln(\tan^{-1} x)$.

4. Find $\frac{dy}{dx}$ for $y = x^2 \sec^{-1} x$, $x > 1$.

19. Challenge ProblemFind every point on the graph of $y = \tan^{-1} x$ where the tangent line is parallel to the line $x - 4y = 7$.

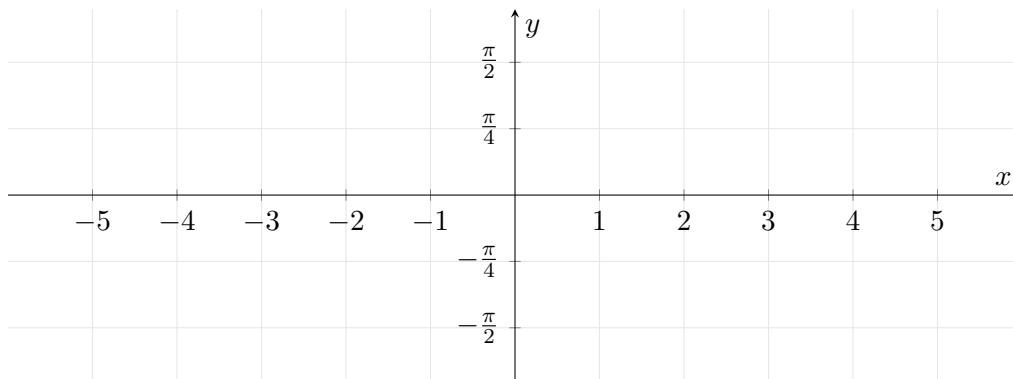
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20. Graphing Challenge

Sketch a function f that satisfies all of the following:

- $f(0) = 0$ and $f'(0) = 1$
- $f'(x) > 0$ for all x
- $f'(x) \rightarrow 0$ as $x \rightarrow \pm\infty$
- $-\frac{\pi}{2} < f(x) < \frac{\pi}{2}$ for all x



1. Which inverse trigonometric function has all of these properties? _____
2. On the same grid, sketch the graph of its derivative. Describe how the two graphs are related.

Reflection: What is one thing you understand better about derivatives of inverse trigonometric functions after completing this worksheet?
