

Name: _____

Date: _____

LIMITS INVOLVING INFINITY

AP Calculus AB Practice Worksheet

Learning Goal: Evaluate limits as $x \rightarrow \pm\infty$ and infinite limits as $x \rightarrow a$, identify horizontal and vertical asymptotes, use the Sandwich Theorem, and find end behavior models for functions.

1. Estimate Limits at Infinity from a Table

Complete the table for

$$f(x) = \frac{2x^2}{x^2 + 1}.$$

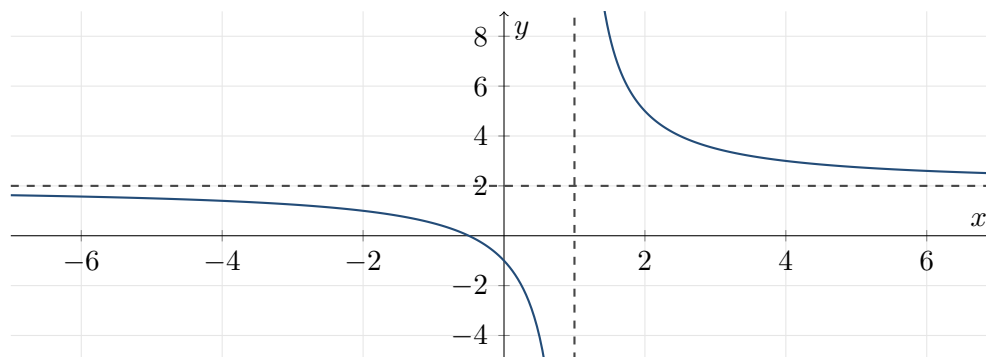
Round to five decimal places.

x	-1000	-100	-10	10	100	1000
$f(x)$						

$\lim_{x \rightarrow -\infty} f(x) = \underline{\hspace{2cm}}$	$\lim_{x \rightarrow \infty} f(x) = \underline{\hspace{2cm}}$
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2. Read Limits from a Graph

The graph of $f(x) = \frac{2x+1}{x-1}$ is shown with its asymptotes (dashed).



1. $\lim_{x \rightarrow \infty} f(x) = \underline{\hspace{2cm}}$

$\lim_{x \rightarrow -\infty} f(x) = \underline{\hspace{2cm}}$

2. $\lim_{x \rightarrow 1^+} f(x) = \underline{\hspace{2cm}}$

$\lim_{x \rightarrow 1^-} f(x) = \underline{\hspace{2cm}}$

Name: _____

Date: _____

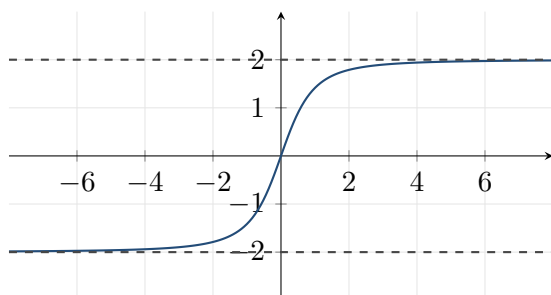
3. Horizontal and Vertical Asymptotes

Horizontal asymptote $y = b$ if $\lim_{x \rightarrow \infty} f(x) = b$ or $\lim_{x \rightarrow -\infty} f(x) = b$.

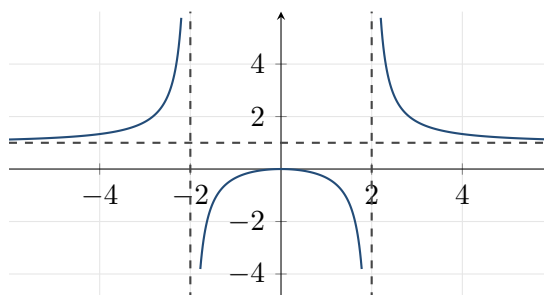
Vertical asymptote $x = a$ if $\lim_{x \rightarrow a^+} f(x) = \pm\infty$ or $\lim_{x \rightarrow a^-} f(x) = \pm\infty$.

4. Read More Limits from Graphs

A: $f(x) = \frac{2x}{\sqrt{x^2 + 1}}$



B: $g(x) = \frac{x^2}{x^2 - 4}$



1. For Graph A: $\lim_{x \rightarrow \infty} f(x) = \underline{\hspace{2cm}}$ $\lim_{x \rightarrow -\infty} f(x) = \underline{\hspace{2cm}}$

List all horizontal asymptotes of f . $\underline{\hspace{4cm}}$

2. For Graph B, complete the table.

$\lim_{x \rightarrow 2^+} g(x)$		$\lim_{x \rightarrow -2^+} g(x)$	
$\lim_{x \rightarrow 2^-} g(x)$		$\lim_{x \rightarrow -2^-} g(x)$	
$\lim_{x \rightarrow \infty} g(x)$		$\lim_{x \rightarrow -\infty} g(x)$	

3. List all asymptotes of g . Is g even, odd, or neither? How does that show up in your table?

Name: _____

Date: _____

5. Rational Functions at Infinity

For $\lim_{x \rightarrow \pm\infty} \frac{p(x)}{q(x)}$, divide every term by the highest power of x in the denominator. Compare degrees:

$\deg p < \deg q$	limit is 0
$\deg p = \deg q$	limit is the ratio of the leading coefficients
$\deg p > \deg q$	limit is $\pm\infty$ (no horizontal asymptote)

6. Evaluate Each Limit

Show the “divide by the highest power” step.

1. $\lim_{x \rightarrow \infty} \frac{5x + 3}{2x - 7}$

2. $\lim_{x \rightarrow -\infty} \frac{4x^2 - x}{x^3 + 2}$

3. $\lim_{x \rightarrow \infty} \frac{3x^3 - x + 1}{6x^3 + 2x^2}$

4. $\lim_{x \rightarrow -\infty} \frac{x^3 - 2x}{5x^2 + 1}$

Name: _____

Date: _____

7. Radicals and Absolute Values at Infinity

Remember: $\sqrt{x^2} = |x|$. So $\sqrt{x^2} = x$ when $x > 0$, but $\sqrt{x^2} = -x$ when $x < 0$. This matters when $x \rightarrow -\infty$.

Evaluate each limit.

1.

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1}}{x} \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 1}}{x}$$

2.

$$\lim_{x \rightarrow -\infty} \frac{3x}{\sqrt{4x^2 + 1}}$$

3.

$$\lim_{x \rightarrow \infty} \left(\sqrt{x^2 + x} - x \right) \quad (\text{Hint: multiply by the conjugate.})$$

4.

$$\lim_{x \rightarrow \infty} \frac{|x| + 3}{2x - 1}$$

Name: _____

Date: _____

8. Exponential, Logarithmic, and Inverse Trig Functions

Evaluate each limit. A quick sketch of the basic graph may help.

1. $\lim_{x \rightarrow \infty} e^{-x} = \underline{\hspace{2cm}}$

$\lim_{x \rightarrow -\infty} e^x = \underline{\hspace{2cm}}$

2. $\lim_{x \rightarrow \infty} \tan^{-1} x = \underline{\hspace{2cm}}$

$\lim_{x \rightarrow -\infty} \tan^{-1} x = \underline{\hspace{2cm}}$

3. $\lim_{x \rightarrow 0^+} \ln x = \underline{\hspace{2cm}}$

$\lim_{x \rightarrow \infty} \ln x = \underline{\hspace{2cm}}$

4. $\lim_{x \rightarrow \infty} \frac{5 + 3e^{-x}}{2 - e^{-2x}}$

5. $\lim_{x \rightarrow \infty} \frac{e^x}{e^x + 1}$ and $\lim_{x \rightarrow -\infty} \frac{e^x}{e^x + 1}$

9. The Sandwich Theorem at Infinity

If $g(x) \leq f(x) \leq h(x)$ for all large x , and $\lim_{x \rightarrow \infty} g(x) = \lim_{x \rightarrow \infty} h(x) = L$, then $\lim_{x \rightarrow \infty} f(x) = L$.

1. Use $-1 \leq \sin x \leq 1$ to show that $\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$.

2. Evaluate $\lim_{x \rightarrow \infty} \frac{2x + \cos x}{x}$.

Name: _____

Date: _____

10. Infinite Limits

Find each one-sided limit. Write ∞ , $-\infty$, or a number. Think about the *sign* of the numerator and the denominator.

1.

$$\lim_{x \rightarrow 3^-} \frac{1}{x-3} \quad \text{and} \quad \lim_{x \rightarrow 3^+} \frac{1}{x-3}$$

2.

$$\lim_{x \rightarrow -1} \frac{x^2}{(x+1)^2}$$

3.

$$\lim_{x \rightarrow 2^+} \frac{x+2}{x^2-4} \quad \text{and} \quad \lim_{x \rightarrow -2} \frac{x+2}{x^2-4}$$

4.

$$\lim_{x \rightarrow (\pi/2)^-} \tan x \quad \text{and} \quad \lim_{x \rightarrow (\pi/2)^+} \tan x$$

5.

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{x}$$

Name: _____

Date: _____

11. Find All Asymptotes

For each function, find every horizontal and vertical asymptote. Justify each one with a limit. Note any holes.

1.

$$f(x) = \frac{3x^2 - 1}{x^2 - x - 6}$$

2.

$$g(x) = \frac{x + 1}{x^2 - 1}$$

3.

$$h(x) = \frac{5e^x}{e^x - 2} \quad (\text{Hint: consider } x \rightarrow \infty \text{ and } x \rightarrow -\infty \text{ separately.})$$

Name: _____

Date: _____

12. End Behavior Models

The function g is a

right end behavior model for f if $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$,
left end behavior model for f if $\lim_{x \rightarrow -\infty} \frac{f(x)}{g(x)} = 1$.

For a polynomial, the leading term is an end behavior model in both directions.

1. Show that $g(x) = 3x^4$ is an end behavior model for $f(x) = 3x^4 - 2x^3 + 5x - 1$.

2. Find a simple end behavior model for

$$f(x) = \frac{3x^4 - 2x^3 + 5}{x^2 + 1}.$$

What does it tell you about $\lim_{x \rightarrow \pm\infty} f(x)$?

3. Find a right end behavior model and a left end behavior model for

$$f(x) = x + e^{-x}.$$

Name: _____

Date: _____

13. “Seeing” Limits with a Substitution

Replacing x with $\frac{1}{t}$ turns a limit as $x \rightarrow \infty$ into a limit as $t \rightarrow 0^+$:

$$\lim_{x \rightarrow \infty} f(x) = \lim_{t \rightarrow 0^+} f\left(\frac{1}{t}\right)$$

1. Use the substitution to find $\lim_{x \rightarrow \infty} x \sin\left(\frac{1}{x}\right)$. (Recall $\lim_{t \rightarrow 0} \frac{\sin t}{t} = 1$.)

2. Use the substitution to find $\lim_{x \rightarrow -\infty} \cos\left(\frac{1}{x}\right)$.

14. Explain Your Thinking

1. Can the graph of a function cross its horizontal asymptote? Use $f(x) = \frac{\sin x}{x}$ to support your answer.

2. Can the graph of a continuous function cross a vertical asymptote? Explain.

3. A student says, “If the denominator of a rational function is 0 at $x = a$, then $x = a$ is a vertical asymptote.” Give a counterexample.

Name: _____

Date: _____

15. Limits in Context

1. The population of fish in a lake is modeled by

$$P(t) = \frac{1200}{1 + 5e^{-0.3t}},$$

where t is in years.

(a) Find the initial population $P(0)$. _____

(b) Find $\lim_{t \rightarrow \infty} P(t)$ and explain what it means.

2. After a dose, the concentration of a drug in the bloodstream is

$$C(t) = \frac{10t}{t^2 + 4} \text{ mg/L},$$

where t is in hours.

(a) Find $\lim_{t \rightarrow \infty} C(t)$ and explain what it means.

(b) Does the graph of C have a horizontal asymptote? A vertical asymptote for $t \geq 0$? Explain.

3. The average cost of producing x units is

$$\overline{C}(x) = \frac{5000 + 12x}{x} \text{ dollars per unit.}$$

Find $\lim_{x \rightarrow \infty} \overline{C}(x)$ and interpret it.

Name: _____

Date: _____

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Mixed Review

16. Mixed Practice

Show your work clearly.

1. Evaluate $\lim_{x \rightarrow \infty} \frac{7 - 2x^2}{3x^2 + x}$.

2. Evaluate $\lim_{x \rightarrow 4^-} \frac{x}{x - 4}$.

3. Evaluate $\lim_{x \rightarrow -\infty} \frac{\sqrt{9x^2 + 2}}{x + 1}$.

4. Find all asymptotes of $y = \frac{2x^2 - 8}{x^2 - 5x + 6}$.

17. Challenge Problem

Find the constants a and b so that the graph of

$$f(x) = \frac{ax + 5}{3 - bx}$$

has a vertical asymptote at $x = 1$ and a horizontal asymptote at $y = -2$.

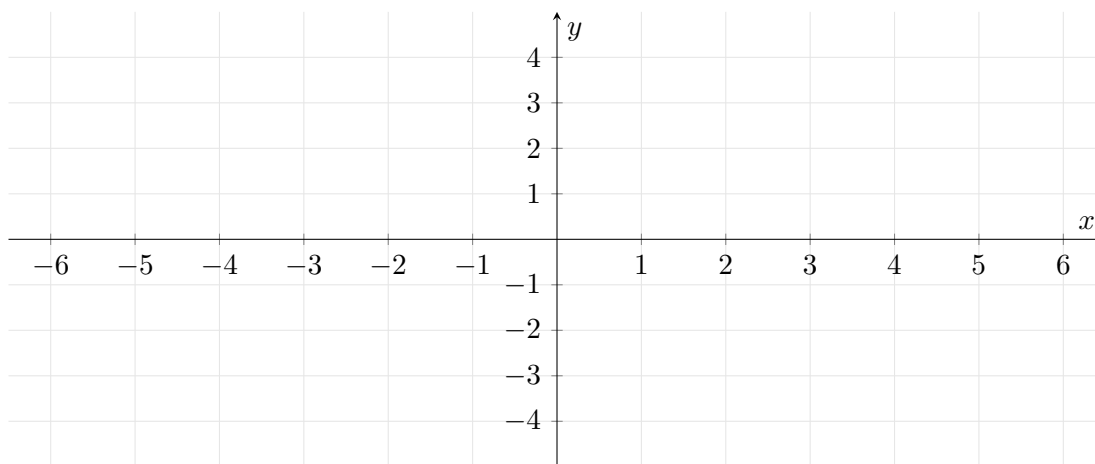
Name: _____

Date: _____

18. Graphing Challenge

Sketch the graph of a function f that satisfies all of the following:

- $\lim_{x \rightarrow \infty} f(x) = 2$ and $\lim_{x \rightarrow -\infty} f(x) = -1$
- $\lim_{x \rightarrow 1^-} f(x) = \infty$ and $\lim_{x \rightarrow 1^+} f(x) = -\infty$
- $f(0) = 0$ and f is continuous everywhere except at $x = 1$



Draw all asymptotes as dashed lines. Then explain how each condition appears on your graph.

Reflection: What is one thing you understand better about limits involving infinity after completing this worksheet?

