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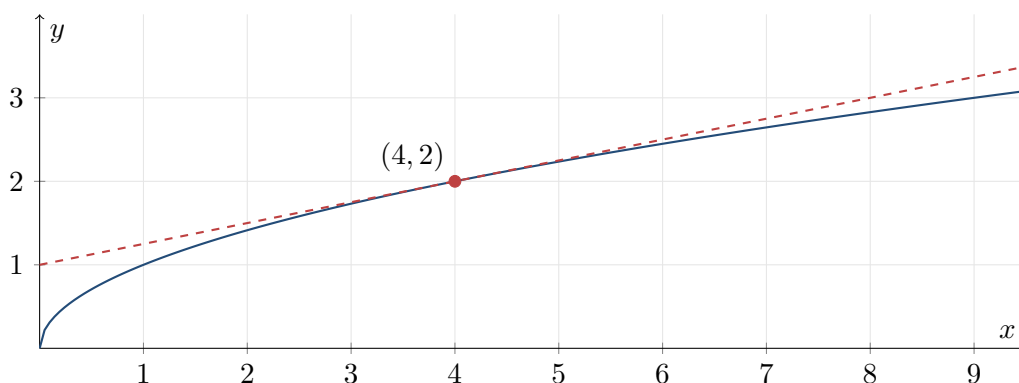
LINEARIZATION AND DIFFERENTIALS

AP Calculus AB Practice Worksheet

Learning Goal: Use tangent lines to approximate function values, decide whether an approximation is an overestimate or underestimate, use differentials to estimate change, and analyze error propagation in measurements.

1. Tangent Lines Approximate Curves

The graph of $f(x) = \sqrt{x}$ is shown with its tangent line (dashed) at $(4, 2)$.



- Find the equation of the tangent line at $x = 4$. _____
- Use the tangent line to estimate $\sqrt{4.1}$ and $\sqrt{8}$. Compare each with a calculator value.

x	Tangent line estimate	Calculator value	Error
4.1			
8			

- Why is one estimate much better than the other?

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2. Linearization

If f is differentiable at $x = a$, the **linearization** of f at a is

$$L(x) = f(a) + f'(a)(x - a)$$

For x near a , $f(x) \approx L(x)$. This is called the *standard linear approximation* of f at a , and a is the *center*.

3. Find the Linearization

Find the linearization $L(x)$ of each function at the given center.

1.

$$f(x) = x^3 - 2x, \quad a = 2$$

2.

$$f(x) = \sqrt{x+3}, \quad a = 1$$

3.

$$f(x) = \sin x, \quad a = \frac{\pi}{6}$$

4.

$$f(x) = \ln x, \quad a = 1$$

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4. Estimate Values with Linearization

Choose a function f and a convenient center a . Then use $L(x)$ to estimate each value. Do not use a calculator until the last column.

Estimate	$f(x)$	a	$L(x)$	Estimate	Calculator
$\sqrt{26}$					
$\sqrt[3]{8.06}$					
$(1.98)^4$					
$\frac{1}{0.98}$					
$e^{0.05}$					

Show the work for any two rows.

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5. Standard Linear Approximations Near $x = 0$

$$(1 + x)^k \approx 1 + kx$$

$$e^x \approx 1 + x$$

$$\sin x \approx x$$

$$\ln(1 + x) \approx x$$

$$\cos x \approx 1$$

$$\tan x \approx x$$

1. Verify the first approximation by finding the linearization of $f(x) = (1 + x)^k$ at $x = 0$.

2. Use the approximations to estimate each value.

$\sqrt{1.02}$		$\sin(0.03)$	
$(1.003)^{-2}$		$\ln(1.04)$	
$\sqrt[3]{0.97}$		$e^{-0.02}$	

3. Find the linearization of each function near $x = 0$.

(a) $\sqrt{1 + 4x}$ _____

(b) $\frac{1}{(1 - x)^3}$ _____

(c) $e^{3x} \cos x$ _____

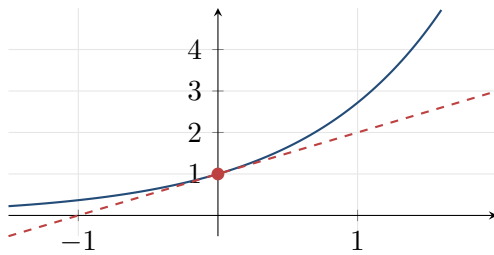
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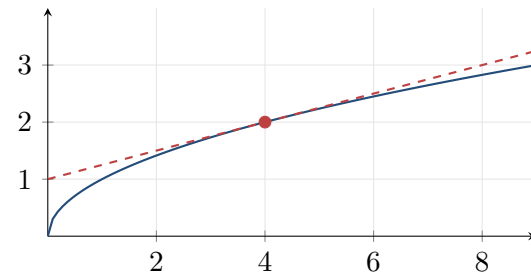
6. Overestimate or Underestimate?

Concavity near a	Tangent line lies	$L(x)$ gives an
Concave up ($f'' > 0$)	below the curve	underestimate
Concave down ($f'' < 0$)	above the curve	overestimate

Concave up: $y = e^x$



Concave down: $y = \sqrt{x}$



For each estimate, find f'' near the center and decide whether $L(x)$ is an overestimate or an underestimate.

- Using the linearization of $f(x) = e^x$ at 0 to estimate $e^{0.1}$.

- Using the linearization of $f(x) = \sqrt{x}$ at 4 to estimate $\sqrt{4.1}$.

- Using the linearization of $f(x) = \ln x$ at 1 to estimate $\ln(1.2)$.

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7. AP-Style Problems: Tangent Line Approximation

1. The function f is twice differentiable. Selected values are shown.

x	2	3	5
$f(x)$	5	4	1
$f'(x)$	-3	-1	2
$f''(x)$	4	3	1

- (a) Write an equation of the line tangent to the graph of f at $x = 2$. Use it to approximate $f(2.1)$.

- (b) Is your approximation an overestimate or an underestimate? Justify. (Assume $f'' > 0$ on $[2, 3]$.)

2. Let $y = f(x)$ be the solution to the differential equation

$$\frac{dy}{dx} = 2x - y \quad \text{with} \quad f(1) = 3.$$

- (a) Write an equation of the tangent line to the graph of f at $x = 1$. Use it to approximate $f(1.2)$.

- (b) Find $\frac{d^2y}{dx^2}$ in terms of x and y . Evaluate it at $(1, 3)$. Is your approximation in part (a) an overestimate or an underestimate? Justify.

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8. Differentials

Let $y = f(x)$. The **differential** dx is an independent variable, and the differential dy is

$$dy = f'(x) dx$$

The actual change is $\Delta y = f(x + dx) - f(x)$. For small dx , $\Delta y \approx dy$.

9. Find the Differential

Find dy .

1. $y = x^5 - 3x^2 + 7$

2. $y = \sin(3x)$

3. $y = \sqrt{1 - x^2}$

4. $y = xe^x$

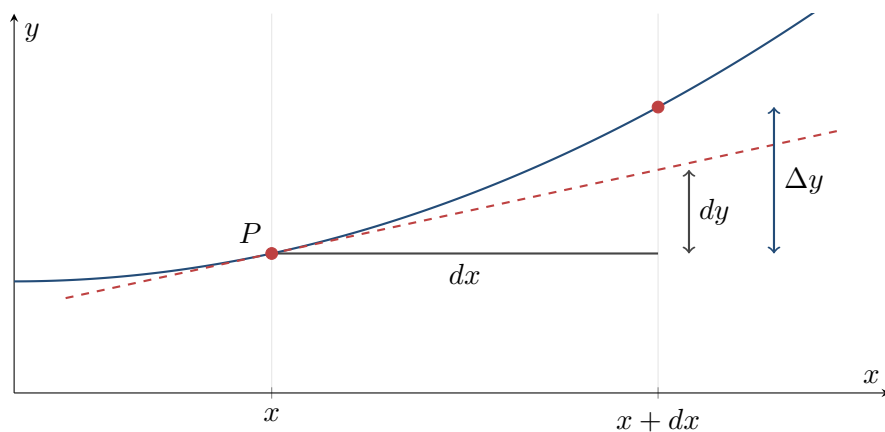
5. Find the value of dy for $y = x^3 - x$ when $x = 2$ and $dx = 0.05$.

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10. Comparing Δy and dy

The diagram shows a curve, its tangent line at P , and the changes dx , dy , and Δy .



1. For $f(x) = x^2$ at $x = 3$, complete the table.

dx	$\Delta y = f(3 + dx) - f(3)$	$dy = f'(3) dx$	$ \Delta y - dy $
0.1			
0.01			
0.001			

2. Show algebraically that $\Delta y - dy = (dx)^2$ for this function. What happens to the error as $dx \rightarrow 0$?

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11. Estimating Change with Differentials

	True	Estimated
Absolute change	$\Delta f = f(a + dx) - f(a)$	$df = f'(a) dx$
Relative change	$\frac{\Delta f}{f(a)}$	$\frac{df}{f(a)}$
Percentage change	$\frac{\Delta f}{f(a)} \times 100\%$	$\frac{df}{f(a)} \times 100\%$

- The radius of a circle increases from $r = 10$ cm to $r = 10.1$ cm. Use dA to estimate the change in area. Compare with the exact change ΔA .

- The radius of a sphere increases from 5 in to 5.02 in. Estimate the change in volume. ($V = \frac{4}{3}\pi r^3$)

- The edge of a cube increases from 8 cm to 8.1 cm. Estimate the percentage change in its surface area.

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12. Sensitivity to Change and Error Propagation

A measured quantity with a small error dx produces an error $dy \approx f'(x) dx$ in any value computed from it.

1. The side of a cube is measured as 10 cm with a possible error of ± 0.05 cm. Estimate the maximum possible error in the computed volume. What is the percentage error?

2. The radius of a circle is measured as 8 cm. How accurately must it be measured so that the error in the computed area is at most 1%?

3. A 12-ft ladder leans against a wall. The angle θ between the ladder and the ground is measured as $\frac{\pi}{3}$ with a possible error of ± 0.02 radians. The height reached is $h = 12 \sin \theta$. Estimate the error in h .

13. Explain Your Thinking

1. Why does a linear approximation usually get worse as x moves farther from the center a ?

2. In Section 10, why is dy different from Δy ? When are they equal?

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LINEARIZATION AND DIFFERENTIALS

Mixed Review

14. Mixed Practice

Show your work clearly.

1. Find the linearization of $f(x) = \tan x$ at $a = \frac{\pi}{4}$ and use it to estimate $\tan(0.8)$.

2. Estimate $\sqrt[4]{15.9}$ using a linearization.

3. Find dy for $y = \frac{x}{x+1}$, then evaluate it at $x = 1$, $dx = 0.1$.

4. The radius of a sphere is measured as $3 \text{ cm} \pm 0.01 \text{ cm}$. Estimate the error in the computed surface area $S = 4\pi r^2$.

15. Challenge Problem

The linearization of f at $x = 3$ is $L(x) = 4x - 5$. Let $g(x) = x f(x)$. Find the linearization of g at $x = 3$ and use it to estimate $g(3.05)$.

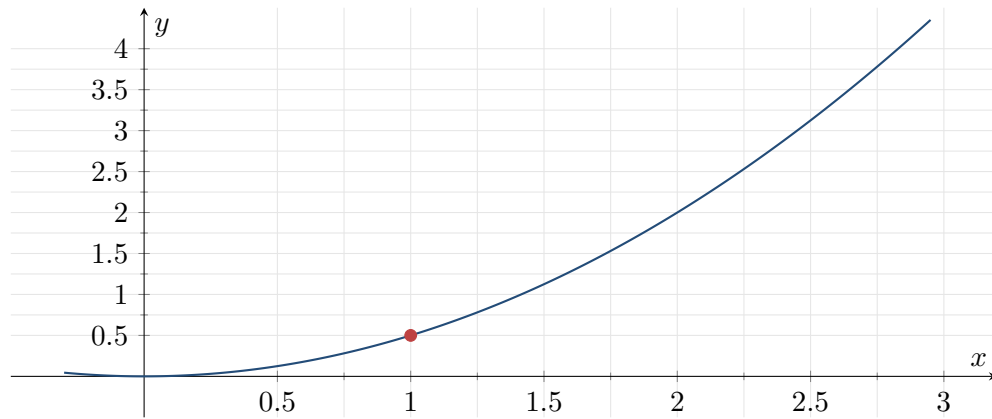
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16. Graphing Challenge

The graph of $f(x) = \frac{1}{2}x^2$ is shown.

- Draw the tangent line to f at $x = 1$.
- Mark $dx = 1$, then show dy and Δy on the graph, as in Section 10.
- Compute dy and Δy and compare.



1. $dy =$ _____ $\Delta y =$ _____

2. Is the tangent line approximation of $f(2)$ an overestimate or an underestimate? Explain using the picture.

Reflection: What is one thing you understand better about linearization and differentials after completing this worksheet?
