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MODELING AND OPTIMIZATION

AP Calculus AB Practice Worksheet

Learning Goal: Translate real-world situations into functions, use derivatives to find maximum and minimum values, justify that a critical point gives the optimal value, and interpret answers in context with correct units.

1. Strategy for Solving Optimization Problems

1. **Understand** the problem. What is given? What must be maximized or minimized?
2. **Draw** a picture and label it with variables.
3. Write the **objective function** (the quantity to optimize).
4. Use a **constraint** equation to write the objective in terms of *one* variable.
5. State the **domain** that makes sense in the problem.
6. Find the **critical points** and test them (and the endpoints, if any).
7. **Justify** your answer and **interpret** it in context, with units.

2. Warm-Up: Explore Before You Differentiate

Find two positive numbers whose sum is 20 and whose product is as large as possible.

1. Complete the table.

First number x	2	6	8	10	12	16
Second number						
Product P						

2. Write P as a function of x , state its domain, and use calculus to confirm the best choice of x .

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3. Number Problems

Identify the objective and the constraint, then solve. Justify that you have found a maximum or minimum.

1. Find two positive numbers whose product is 64 and whose sum is as small as possible.

2. Find the number that exceeds its square by the greatest amount.

3. Find the positive number x for which the sum of x and its reciprocal is as small as possible.

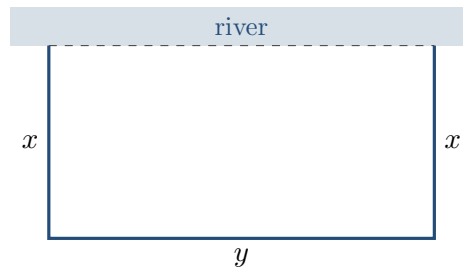
4. Find two numbers whose difference is 10 and whose product is as small as possible.

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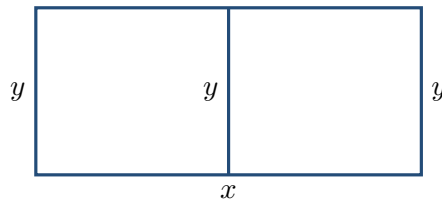
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4. Fencing Problems

1. A farmer has 2400 ft of fencing to enclose a rectangular field that borders a straight river. No fence is needed along the river. What dimensions give the largest area?



2. A rancher wants to enclose 1500 m^2 in a rectangular pen and divide it into two smaller pens with one fence parallel to one side. What dimensions use the least total fencing?

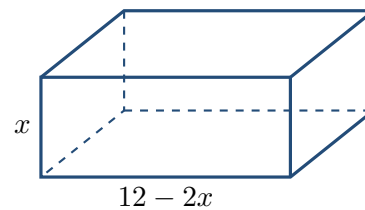
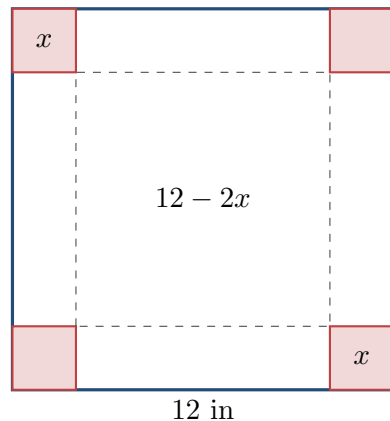


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5. The Open-Box Problem

An open-top box is made from a 12-inch by 12-inch square of cardboard by cutting equal squares of side x from each corner and folding up the sides.



1. Write the volume V as a function of x . What is the domain?

2. Find the value of x that maximizes the volume, and the maximum volume.

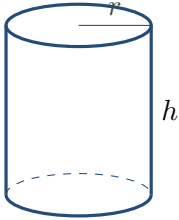
3. Justify that your answer is a maximum, using the Candidates Test on the domain.

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6. Designing a Can

A cylindrical can must hold 1000 cm^3 . Find the radius and height that use the least metal (minimum total surface area, including the top and bottom).



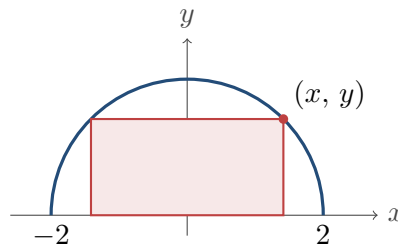
$$\text{Volume: } V = \pi r^2 h$$

$$\text{Surface area: } S = 2\pi r^2 + 2\pi r h$$

What is the relationship between the height and the radius of the best can?

7. Inscribed Rectangle

A rectangle is inscribed in a semicircle of radius 2, with its base on the diameter. Find the largest possible area of the rectangle.

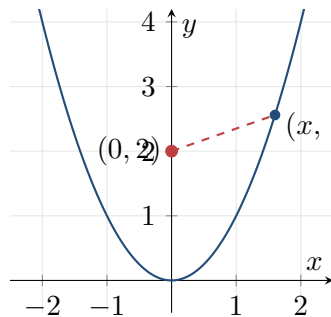


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8. Closest Point on a Curve**Tip:** Minimizing the distance d is the same as minimizing d^2 , which avoids the square root.

1. Find the points on the parabola $y = x^2$ that are closest to the point $(0, 2)$.



2. Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$.

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9. Optimization in Economics**Profit**

$$P(x) = R(x) - C(x) \quad (\text{revenue minus cost})$$

Maximum profit

$$\text{occurs where } R'(x) = C'(x)$$

$$(\text{marginal revenue} = \text{marginal cost})$$

Minimum average cost

$$\text{occurs where average cost} = \text{marginal cost}$$

1. Suppose $R(x) = 9x$ and $C(x) = x^3 - 6x^2 + 15x$, where x is measured in thousands of units. Is there a production level that maximizes profit? If so, what is it?

2. For the same cost function, find the production level that minimizes the average cost $\frac{C(x)}{x}$.

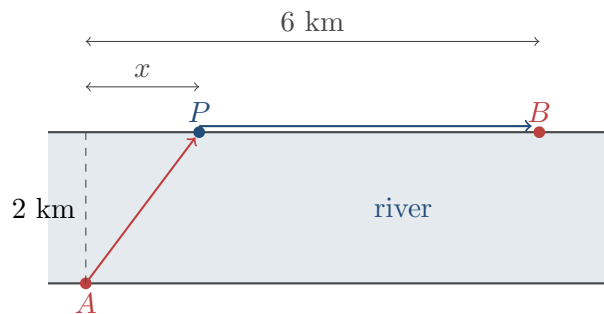
3. A theater sells 500 tickets per show at \$20 each. For every \$1 decrease in price, it sells 50 more tickets. What ticket price maximizes revenue? What is the maximum revenue?

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10. Minimizing Time

A woman at point A on the bank of a straight river 2 km wide wants to reach point B , 6 km downstream on the opposite bank. She can row at 3 km/h and walk at 5 km/h. She rows to a point P on the opposite bank, x km downstream, and then walks to B .



1. Write the total time $T(x)$ for the trip. State the domain.

2. Find the value of x that minimizes the total time. Justify that it is a minimum.

3. Compare with rowing straight to B ($x = 6$) and with rowing straight across ($x = 0$). How much time does the best route save?

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11. Justifying an Optimum

On the AP Exam, you must **justify** that a critical point gives a maximum or minimum.

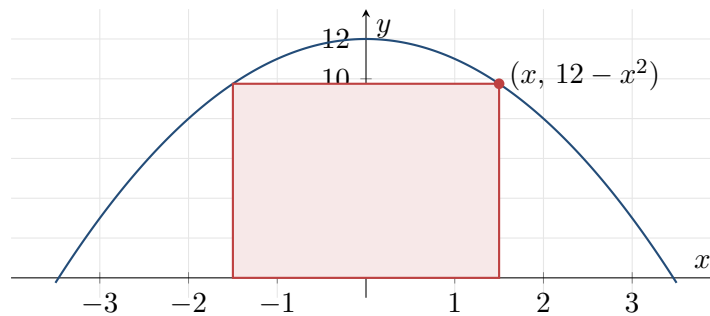
First Derivative Test f' changes from + to - (max) or - to + (min).

Second Derivative Test $f'(c) = 0$ and $f''(c) < 0$ (max) or $f''(c) > 0$ (min).

Candidates Test On a closed interval, compare f at critical points and endpoints.

12. AP-Style Problem

A rectangle has its base on the x -axis and its upper two vertices on the parabola $y = 12 - x^2$.



1. Write the area A of the rectangle as a function of x , for $x > 0$.

2. Find the dimensions of the rectangle with the greatest area.

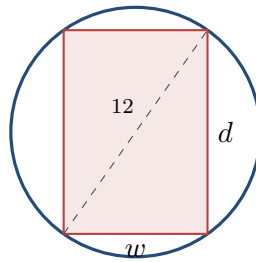
3. Use the Second Derivative Test to justify that your answer gives a maximum.

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13. More Modeling Problems

1. **Strongest beam.** The strength of a rectangular beam is $S = k w d^2$, where w is the width, d is the depth, and k is a positive constant. Find the dimensions of the strongest beam that can be cut from a round log of diameter 12 inches. (Hint: $w^2 + d^2 = 144$.)



2. **Window design.** A Norman window is a rectangle topped by a semicircle. If the perimeter is 20 ft, find the dimensions that admit the most light (maximum area).

14. Explain Your Thinking

1. Why is it important to state the domain of the objective function?

2. A student finds one critical point and immediately calls it the maximum. What should the student have checked?

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MODELING AND OPTIMIZATION

Mixed Review

15. Mixed Practice

Show your work clearly and justify each optimum.

1. A rectangle has perimeter 40 cm. What is its largest possible area?

2. A closed box with a square base must have volume 32 ft^3 . Find the dimensions that minimize the surface area.

3. Find the point on the line $y = 2x + 3$ that is closest to the origin.

16. Challenge Problem

Find the dimensions of the rectangle of largest area that can be inscribed in the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, with sides parallel to the axes. What is the maximum area?

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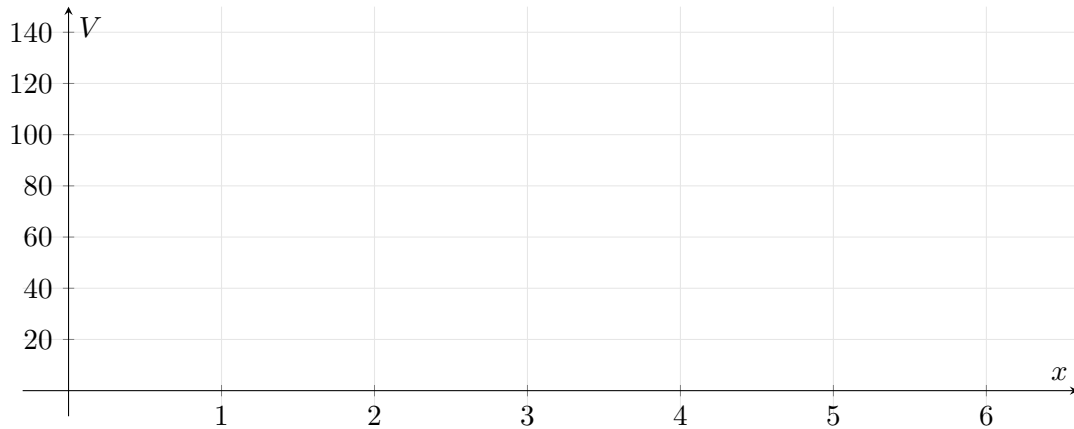
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17. Graphing Challenge

Graph the volume function from the open-box problem (Section 5),

$$V(x) = x(12 - 2x)^2,$$

on its domain.



1. Plot V at $x = 0, 1, 2, 3, 4, 5, 6$, then sketch the curve. Mark the maximum point.

2. What happens to the volume at the endpoints $x = 0$ and $x = 6$? Explain physically.

Reflection: What is one thing you understand better about modeling and optimization after completing this worksheet?
